Longitudinal evolution of a phonon pulse in liquid ^{4}He

2005 J. Phys.: Condens. Matter 17 2859

(http://iopscience.iop.org/0953-8984/17/19/004)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **17** (2005) 2859–2871 doi:10.1088/0953-8984/17/19/004

Longitudinal evolution of a phonon pulse in liquid ^{4}He

I N Adamenko[1,2], **K E Nemchenko**[1], **V A Slipko**[1] **and A F G Wyatt**[3]

[1] Karazin Kharkov National University, Svobody Sq. 4, Kharkov, 61077, Ukraine
[2] Electro-physical Scientific and Technical Centre, National Academy of Sciences of Ukraine, Chernyshevsky Street 28, Kharkov, 61002, Ukraine
[3] School of Physics, University of Exeter, Exeter EX4 4QL, UK

Received 23 February 2005
Published 29 April 2005
Online at stacks.iop.org/JPhysCM/17/2859

Abstract
The longitudinal evolution of a low energy phonon pulse in superfluid helium ^{4}He is analysed theoretically. The phonons are considered to be strongly interacting and are described by the exact local equilibrium distribution function. The theory describes how the longitudinal pulse shape develops. We find that the temperature change in the pulse is determined by a simple running wave which moves with a velocity very close that of the sound velocity. This results in only a small variation of the pulse length as the pulse propagates. This is in contrast to the case of a non-interacting phonon pulse, which shows considerable dispersion. The details of the phonon pulse deformation depend on whether the pulse contains phonons with relatively high energies or not. The higher energy phonons cause the single wave to eventually break at both the front and back of the pulse. Without these high energy phonons, the pulse only breaks at the front. We find that, in the first approximation with respect to the small parameters of the problem, the anisotropic nature of the phonon pulse is conserved as it propagates.

1. Introduction

It has been a long standing problem that a short pulse of phonons in liquid helium, at zero bar, propagates at a constant velocity and does not show the dispersion of velocities that might be expected from the measured non-linear dispersion curve. Moreover, the velocity is that of the usual sound velocity. The variation in the group velocity, derived from the measured $\epsilon(p)$, is $\sim$10% [1] and this would lead to a similar variation in propagation times. Over 15 mm, the propagation time for a short pulse, say 100 ns, would vary between 57 and 63 μs.

In this paper we analyse the propagation of a strongly interacting pulse of phonons. We show that this is the correct description for low energy phonons in liquid helium at 0 bar, because of the strong three-phonon scattering of these phonons. We then find that these phonons propagate at a single velocity without dispersion, and that this velocity is very close to the ultrasonic velocity. The length of the pulse only changes a little. However, the analysis shows that the pulse shape changes on propagation and can cause the pulse to break. This is a slow change, but it should be detectable, after a propagation distance of 15 mm, with a fast detector.

The analysis raises an interesting question concerning the interaction rate of phonons in the energy range 8.5–10.0 K. These phonons cannot interact by a three-phonon process but can interact by non-number conserving multiphonon processes. If the interaction rate is fast enough, these phonons should be included in the interacting pulse. However, this radically changes the phonon distribution function compared to the case where they are not included, but only causes a small change in the velocity of the pulse. We suggest ways to decide whether these phonons should be included or not.

The thermodynamic and kinetic properties of isotropic and weakly anisotropic phonon systems in liquid ^{4}He have been studied since the pioneering work of Landau [2]. But in liquid ^{4}He it turns out to be possible to create a strongly anisotropic quasiparticle systems such as phonon pulses [3–6]. Strongly anisotropic phonon systems can be created by a short ($t_p < 10^{-6}$ s) current pulse in a thin film metal heater immersed in superfluid ^{4}He. The total momentum of such a phonon system is directed along the perpendicular to the heater surface, and the phonon distribution function has a strong anisotropy in momentum space. At first, the phonon pulse is localized near the heater, with transverse dimensions similar to that of the heater, and a longitudinal dimension l_p determined by duration of the heater pulse t_p, i.e. $l_p = ct_p$ where c is the sound velocity in ^{4}He, $c = 238$ m s^{-1}. At a low temperature ($T < 0.05$ K) the thermal excitations in ^{4}He can be neglected and the evolution of a phonon pulse is determined by the strong and anisotropic interactions and the phonon energy–momentum relationship. The energy ε can be written as

$$\varepsilon = cp\,(1 + \psi(p))\,, \tag{1}$$

where p is the phonon momentum. The function $\psi(p)$, in spite of its smallness, determines the possible mechanisms of phonon relaxation [7, 8].

Phonons with momentum $p < p_c^{(\infty)}$, where the superscript denotes the number of phonons created in a spontaneous decay process, are known as ℓ-phonons. When the function $\psi(p)$ is positive, a phonon can decay to many phonons. However phonons with momentum $p > p_c^{(\infty)}$ (h-phonons) are stable against spontaneous decay, because the function $\psi(p)$ is negative in this region of momentum. At the saturated vapour pressure (SVP), $cp_c^{(\infty)}/k_B = 10$ K [9, 10].

The decay threshold $p_c^{(n)}$ depends on n, the number of phonons released in spontaneous decay. At the SVP, the three-phonon threshold momentum is $cp_c^{(2)}/k_B = 8.5$ K [9]. The three-phonon process (3pp) rate has been calculated for both isotropic and anisotropic phonon systems over the whole region of allowed momentum [11]. The rates for such multiphonon processes ($n > 2$) have not been calculated. But these rates are expected, both from theoretical consideration [12] and from the experimental data [13], to be smaller than the 3pp rate.

According to [11] the characteristic time τ_{3pp} for three-phonon processes is much less than all other times in the problem, in particular $\tau_{3pp} \ll t_p$. Therefore, phonons with momenta $p < p_c^{(2)}$ attain local equilibrium practically instantaneously. For phonons with momenta, in the range $p_c^{(2)} < p < p_c^{(\infty)}$ we consider both possibilities; first when all ℓ-phonons are in equilibrium and secondly when only phonons with momenta $p < p_c^{(2)}$ are in equilibrium.

For phonons with momenta $p > p_c^{(\infty)}$, the quickest process is the four-phonon process (4pp) in which the number of phonons is conserved. The characteristic time τ_{4pp} for such processes has been calculated in [14]. It turns out that these h-phonons form a very weakly interacting phonon system of practically ballistic phonons.

The phenomenon of the creation of h-phonons by the ℓ-phonon pulse and its subsequent evolution are described in [15, 16]. In this paper we consider the evolution of the ℓ-phonon pulse and neglect the creation of h-phonons. It is always possible to create an ℓ-phonon pulse of sufficiently low temperature that the creation of h-phonons is negligible [16].

2. Quasiequilibrium distribution function of the anisotropic phonon system

After a time of order of magnitude τ_{3pp}, which is much smaller than the other characteristic times of the problem, the local equilibrium distribution function, for phonons with momenta less than $p_c^{(2)}$, becomes the Bose–Einstein distribution function, involving the drift velocity $\mathbf{u}$ and temperature T:

$$n_u = \left[\exp\left(\frac{\varepsilon - \mathbf{pu}}{k_B T}\right) - 1\right]^{-1}. \quad (2)$$

Usually phonon systems are considered to be practically isotropic which corresponds to small drift velocities, $u \ll c$. For strongly anisotropic phonon systems, created in experiments [3–6], u is of the order of magnitude of c. In this case it is possible to rewrite the local equilibrium distribution function (2) for the anisotropic phonon system, using (1), as follows:

$$n_u = \left[\exp\left(\frac{cp}{k_B T}(\chi + \psi(p) + \zeta(1-\chi))\right) - 1\right]^{-1}, \quad (3)$$

where $\zeta = 1 - \cos\theta$; θ is the angle between the momentum of phonon $\mathbf{p}$ and the direction of the drift velocity $\mathbf{s} = \mathbf{u}/u$, and $\chi = 1 - u/c$.

The function $\psi(p)$, which characterizes the relative deviation of the phonon energy–momentum relation from a linear dependence, can be written in the form [14]

$$\psi(p) = \gamma\left(\frac{p}{p_c^{(\infty)}}\right)^2\left[1 - \left(\frac{p}{p_c^{(\infty)}}\right)^2\right], \quad (4)$$

where $\gamma = 0.185$.

From the necessary condition that $n_u \geqslant 0$, we see from (2) that $u \leqslant \epsilon/p$. So for a system which only has phonons with $p < p_c^{(\infty)}$ with no h-phonons or rotons, see [17], we have $u < c$. For strongly anisotropic phonon systems when the value of parameter χ is an order of the magnitude smaller than the function $\psi(p)$, the distribution function of phonons (3) depends essentially on the function $\psi(p)$, i.e. on the deviation of the phonon dispersion from linearity. As $\psi(p)$ decreases at large momenta, there may appear additional maximum in the phonon energy distribution at large momenta, if the temperature is high enough.

Figure 1 shows the distribution of the phonon energy as a function of phonon momentum for a strongly anisotropic phonon system with parameters $\chi = 0.02$ and $T = 0.04$ K at $\zeta = 0$ (solid line) and at $\zeta = 0.01$ (dashed line). The chosen typical values of parameters χ and T for the strongly anisotropic phonon system correspond to the same energy and momentum densities as in the cone approximation, when a cone is cut out of the isotropic Bose–Einstein distribution in momentum space. The cone angle is taken to be equal to the characteristic angle of the 3pp scattering, $\zeta_p = 0.02$, and the temperature of the isotropic Bose–Einstein distribution is $T_p \sim 1$ K. Phonons emitted by a solid heater rapidly relax and form an anisotropic phonon system which can be approximately described by this cone distribution. This can be taken as the initial condition for the kinetic equation. As a result of phonon–phonon interactions, phonons with momenta $p < p_c^{(\infty)}$ have a local equilibrium with distribution (3). This has the same energy and momentum densities as the injected distribution because phonon–phonon interactions conserve these quantities.

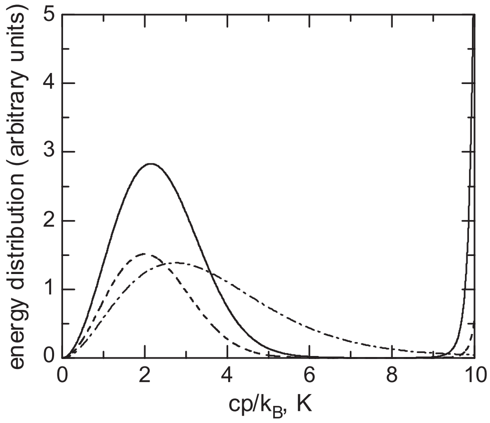

Figure 1. The distribution of the phonon energy as a function of phonon momentum p for a strongly anisotropic phonon system with parameters $\chi = 0.02$ and $T = 0.04$ K in directions $\zeta = 0$ (solid line) and $\zeta = 0.01$ (dashed line). To compare, the dash–dotted line represents the energy distribution for the isotropic Bose–Einstein distribution with temperature $T = 1$ K.

In a strongly anisotropic phonon system, the angular distribution of the phonon energy is characterized by a strong maximum in the forward direction. This can be seen by comparing the distribution of the phonon energy as a function of phonon momentum at $\zeta = 0.00$ and 0.01 in figure 1 (the solid and dashed lines respectively). It should be noted that the group of phonons with momenta close to $p_c^{(\infty)}$ is characterized by sharper angular distribution than the rest of the phonons.

The momentum distribution of phonons in the cone approximation, with a Bose–Einstein distribution, with $\zeta_p = 0.02$ and $T_p = 1$ K, is shown in figure 1, dash–dotted line. We see that this distribution differs markedly from figure 1, solid line, especially at high phonon momenta.

3. The equations that describe the evolution of the anisotropic phonon pulse

We now derive the equations that describe the evolution of a phonon pulse in the superfluid vacuum of ^{4}He. They are determined by the equations which express the conservation of energy and momentum which give the following continuity equations:

$$\frac{\partial E(\mathbf{r},t)}{\partial t} + \frac{\partial \mathbf{Q}_E(\mathbf{r},t)}{\partial \mathbf{r}} = 0, \qquad (5)$$

$$\frac{\partial P_i(\mathbf{r},t)}{\partial t} + \frac{\partial Q_{ij}(\mathbf{r},t)}{\partial r_j} = 0 \qquad (6)$$

where $E(\mathbf{r},t)$ and $\mathbf{Q}_E(\mathbf{r},t)$ are the phonon energy density and energy flux respectively, and $P_i(\mathbf{r},t)$ and $Q_{ij}(\mathbf{r},t)$ are the phonon momentum density and momentum flux respectively.

The phonon energy density $E(\mathbf{r},t)$ is determined by the solution of the kinetic equation for the phonon distribution function $n = n(\mathbf{p},\mathbf{r},t)$ as follows:

$$E(\mathbf{r},t) = \int \mathrm{d}\Gamma\, \varepsilon n; \qquad (7)$$

the vector of the energy density flux is

$$\mathbf{Q}_E(\mathbf{r},t) = \int \mathrm{d}\Gamma\, \varepsilon \frac{\partial \varepsilon}{\partial \mathbf{p}} n; \qquad (8)$$

the phonon momentum density is

$$\mathbf{P}(\mathbf{r}, t) = \int \mathrm{d}\Gamma\, \mathbf{p} n; \tag{9}$$

and the tensor of phonon momentum density flux is

$$Q_{ij}(\mathbf{r}, t) = \int \mathrm{d}\Gamma\, p_i \frac{\partial \varepsilon}{\partial p_j} n, \tag{10}$$

where $\mathrm{d}\Gamma = \mathrm{d}^3 p/(2\pi\hbar)^3$.

In the approximation of instant ℓ-phonon relaxation, and neglecting all dissipative processes, the local equilibrium distribution function n, appearing in equations (7)–(10), is the distribution function of the anisotropic phonon system, n_u, equation (3), which makes the collision integral for phonons equal to zero.

Substituting equation (1) into expression equation (7) we obtain for the phonon energy density

$$E = [1 + \langle \psi(p) \rangle]\, E_0, \tag{11}$$

where we write

$$E_0 = \int \mathrm{d}\Gamma\, cpn_u, \tag{12}$$

and the energy average of an arbitrary function $F = F(\mathbf{p}, \mathbf{r}, t)$ is defined as

$$\langle F \rangle = \int \mathrm{d}\Gamma\, cpn_u F/E_0. \tag{13}$$

As the total phonon momentum is directed along the drift velocity $\mathbf{u}$, we find from equation (9)

$$\mathbf{P} = [1 - \langle \zeta \rangle]\, \mathbf{s} E_0/c. \tag{14}$$

Substituting equation (1) into (8) and taking into account that the total energy density flux is also directed along the drift velocity $\mathbf{u}$, and neglecting the squared terms with respect to the small parameters $\langle \psi(p) \rangle$ and $\langle \zeta \rangle$, we obtain an equation for the energy density flux:

$$\mathbf{Q}_E = \left[1 + \langle \psi(p) \rangle + \left\langle (p\psi(p))' \right\rangle - \langle \zeta \rangle\right] \mathbf{s} E_0 c \tag{15}$$

where the prime denotes differentiation with respect to p. Taking into consideration the symmetry of equation (10), we write the general expression for the tensor of phonon momentum density flux as

$$Q_{ij} = A\delta_{ij} + Bs_i s_j, \tag{16}$$

where A and B are scalar quantities.

To determine these quantities, we multiply equation (16) by the Kronecker function δ_{ij} and sum over indexes, and find

$$3A + B = Q_{ii}. \tag{17}$$

By multiplying equation (16) by $s_i s_j$ and summing over indexes, we find the second relation

$$A + B = Q_{ij} s_i s_j. \tag{18}$$

Solving the system of equations (17) and (18) we have

$$A = \tfrac{1}{2}(Q_{ii} - Q_{ij} s_i s_j), \tag{19}$$

$$B = \tfrac{1}{2}(3Q_{ij} s_i s_j - Q_{ii}). \tag{20}$$

By calculating the sums in equations (19) and (20) in the same approximation as was used for the energy density flux, we obtain the following expression for the tensor of phonon momentum density flux:

$$Q_{ij} = \langle\zeta\rangle E_0\delta_{ij} + \left[1 + \left\langle(p\psi(p))'\right\rangle - 3\langle\zeta\rangle\right] E_0 s_i s_j. \tag{21}$$

Using the local equilibrium distribution function (3) it is possible to integrate explicitly over the angular variables in equation (12). For strongly anisotropic phonon systems, we can neglect the exponentially small number of phonons which move backwards, and so substituting $\zeta_{\max} = \infty$ for $\zeta_{\max} = 2$, we obtain

$$E_0 = -\frac{k_{\mathrm{B}}T}{4\pi^2\hbar^3(1-\chi)}\int_0^{p_{\mathrm{c}}} \mathrm{d}p\, p^2 \ln\left[1 - \exp\left(-\frac{cp}{k_{\mathrm{B}}T}(\chi + \psi(p))\right)\right]. \tag{22}$$

Using equations (13) and (22), we obtain an expression for the average of any function which depends on the modulus of momentum, $f = f(p)$:

$$\langle f(p)\rangle_{p_{\mathrm{c}}} = \frac{\int_0^{p_{\mathrm{c}}} \mathrm{d}p\, p^2 f(p) \ln\left[1 - \exp\left(-\frac{cp}{k_{\mathrm{B}}T}(\chi + \psi(p))\right)\right]}{\int_0^{p_{\mathrm{c}}} \mathrm{d}p\, p^2 \ln\left[1 - \exp\left(-\frac{cp}{k_{\mathrm{B}}T}(\chi + \psi(p))\right)\right]}. \tag{23}$$

Here the subscript p_{c}, on average, corresponds to the upper limit of integration over momentum p, for those phonons which are in local equilibrium with the distribution function (3). We will use this notation when we need to emphasize the dependence of the average on some critical momentum p_{c}.

The expressions in equations (22) and (23) are determined by the dependence of the phonon energy density E and the averages $\langle\psi(p)\rangle$ and $\langle(p\psi(p))'\rangle$ over the parameters χ and T of the local equilibrium distribution function of the anisotropic phonon system.

The average $\langle\zeta\rangle$, which characterizes the angular width of the anisotropic phonon system, is found by carrying out the integration over the angular variables in equation (13). It can be expressed through the value χ and the average $\langle(p\psi(p))'\rangle$:

$$\langle\zeta\rangle_{p_{\mathrm{c}}^{(2)}} = \langle\zeta\rangle_{p_{\mathrm{c}}^{(\infty)}} = \frac{1}{2(1-\chi)}\left\{\chi + \langle(p\psi(p))'\rangle_{p_{\mathrm{c}}^{(2)}}\right\}. \tag{24}$$

For the limiting case of small temperatures T and not very small χ in equations (22) and (23), we can neglect the dispersion as the deviation $\psi(p)$ is in the exponent. Then it is possible to integrate over momentum p explicitly by substituting infinity for the upper limit of integration p_{c}. We find

$$E = \frac{\pi^2 k_{\mathrm{B}}^4 T^4}{180\hbar^3 c^3 (1-\chi)\chi^3}, \tag{25}$$

$$\langle(p\psi(p))'\rangle = 3\langle\psi(p)\rangle = \frac{36\gamma k_{\mathrm{B}}^2 T^2}{c^2 p_{\mathrm{c}}^2 \chi^2}. \tag{26}$$

These approximate formulae are valid if the average $\langle\psi(p)\rangle$, given by equation (26), is small with respect to value χ. For this limiting case, in accordance with equation (25), the energy density of an extremely anisotropic phonon system depends strongly on the value of χ, which is determined by the angular width of the phonon pulse (see equation (24)). The energy density increases with decreasing χ. Therefore small values of χ, which correspond to strongly anisotropic phonon systems, require low temperatures T, compared to that for an isotropic phonon system, to obtain the same energy density.

We see from figure 1 that the phonon quasiequilibrium distribution function for the strongly anisotropic phonon system (3) is almost equal to zero at high momenta. However for the small region near the critical momentum $p_{\mathrm{c}}^{(\infty)}$, where $\psi(p)$ approaches zero, the distribution function

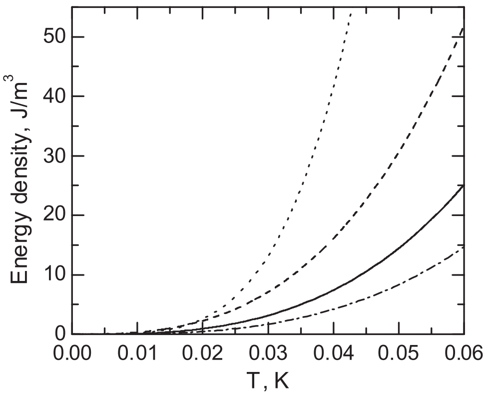

Figure 2. The dependence of the energy density E for the strongly anisotropic phonon system on temperature T at different values $\chi = 0.01$ (dashed line) $\chi = 0.02$ (solid line) and $\chi = 0.03$ (dash–dotted line). The dotted line represents the asymptotic T^4 behaviour for $\chi = 0.03$.

can be large. The contribution of these phonons to the energy can be approximately calculated by using the fact that the distribution function n_u, equation (3), changes rapidly near $p_c^{(\infty)}$.

By integrating over p in equation (22) we obtain for the addition to the phonon energy density

$$\Delta E = E_{p_c^{(\infty)}} - E_{p_c^{(2)}} = \frac{(p_c^{(\infty)})^2 k_B^2 T^2}{4\pi^2 c\hbar^3 (1-\chi)(p_c^{(\infty)}|\psi'_{p_c^{(\infty)}}| - \chi)} \exp\left(-\frac{c p_c^{(\infty)} \chi}{T}\right). \tag{27}$$

Similarly, integrating over p in equation (23) we find the contribution of the phonons with momenta close to $p_c^{(\infty)}$, to the average of $\langle (p\psi(p))' \rangle$:

$$\Delta\langle (p\psi(p))' \rangle = \langle (p\psi(p))' \rangle_{p_c^{(\infty)}} - \langle (p\psi(p))' \rangle_{p_c^{(2)}} = -p_c^{(\infty)} |\psi'_{p_c^{(\infty)}}| \frac{\Delta E}{E}. \tag{28}$$

The average $\langle \psi(p) \rangle$ is not affected by phonons with momenta close to $p_c^{(\infty)}$ because ψ_p approaches zero at $p = p_c^{(\infty)}$. The contribution of these phonons to $\langle \zeta \rangle$ can also be neglected, because from equation (24), phonons with momenta close to $p_c^{(\infty)}$ are concentrated within a small range of angles so they do not contribute appreciably to the angular average $\langle \zeta \rangle$.

It should be noted that the expressions (27) and (28), describing the contribution of phonons with momenta close to $p_c^{(\infty)}$ to the energy density E and the average $\langle (p\psi(p))' \rangle$, grow exponentially with increasing temperature T and decreasing χ.

Figure 2 shows the dependence of the energy density E for the strongly anisotropic phonon system on temperature T at different values $\chi = 0.01$ (dashed line), $\chi = 0.02$ (solid line) and $\chi = 0.03$ (dot–dashed line) calculated from equation (22). When the temperature T increases from very small values, the energy density E increases as T^4 (see equation (25), dotted line). At higher values of T, the energy density grows much more slowly with T, due to the effect of temperature T in the exponent of equation (22).

Figure 3 shows the temperature dependence of the averages $\langle (p\psi(p))' \rangle_{p_c^{(2)}}$ (solid line), $\langle (p\psi(p))' \rangle_{p_c^{(\infty)}}$ (dashed line) and $3\langle \psi(p) \rangle$ (dash–dotted line) at $\chi = 0.02$. When the temperature T increases from zero, all these averages increase as T^2 in accordance with the limiting case equation (26). At higher temperatures, the averages increase much more slowly for the same reason as for the energy density E. At large enough temperatures, the average $\langle (p\psi(p))' \rangle_{p_c^{(\infty)}}$ reaches a maximum after which it decreases, deviating from the average $\langle (p\psi(p))' \rangle_{p_c^{(2)}}$ due to the negative value of $\Delta\langle (p\psi(p))' \rangle$, described by equation (28).

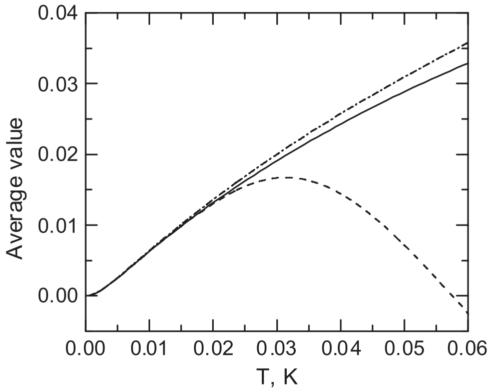

Figure 3. The dependence on the temperature T of the averages $\langle (p\psi(p))' \rangle_{p_c^{(2)}}$ (solid line), $\langle (p\psi(p))' \rangle_{p_c^{(\infty)}}$ (dashed line) and $3\langle \psi(p) \rangle$ (dash–dotted line) at $\chi = 0.02$.

The average $\langle \zeta \rangle$, which characterizes the angular width of the strongly anisotropic phonon system, in the range of T and χ under consideration, in accordance with equation (24), increases monotonically both with increasing temperature T and χ.

4. Longitudinal evolution of the phonon pulse

In this section we develop equations which enable us to calculate the longitudinal evolution of the phonon pulse. In the experiment [6] the phonon system, created in the superfluid ^{4}He, has transverse dimensions much larger than the longitudinal dimension. These are determined by the heater dimensions and the heater pulse duration respectively. It is of interest to consider the pulse evolution, determined by the spatial non-uniformity of the local equilibrium values T and χ in the direction of pulse propagation, the axis z.

We consider the case when all the values which characterize the pulse depend only on the coordinate z and time t. The drift velocity $\mathbf{u}$ is initially directed along the axis z. The x- and y-components of momentum (6) are conserved identically, if $s_x = s_y \equiv 0$, i.e. the drift velocity remains directed along the axis z. The laws of conservation of energy (5) and z-component of momentum (6), taking into account equations (14), (15) and (21), have the form

$$\frac{\partial E}{c\partial t} + \frac{\partial}{\partial z}\left[\left(1 + \langle (p\psi(p))' \rangle - \langle \zeta \rangle\right) E\right] = 0, \tag{29}$$

$$\frac{\partial}{c\partial t}\left[(1 - \langle \psi(p) \rangle - \langle \zeta \rangle)\, E\right] + \frac{\partial}{\partial z}\left[\left(1 + \langle (p\psi(p))' \rangle - 2\langle \zeta \rangle - \langle \psi(p) \rangle\right) E\right] = 0, \tag{30}$$

where we have eliminated E_0, using equation (11), from the expressions for the momentum density (14), and the fluxes (15) and (21).

The system of equations (29) and (30) should be accompanied by the expressions (11), (22)–(24) and (28) which determine the dependence of the phonon energy density and the averages on the independent values χ and T. These relations, combined with the initial conditions for the values of χ and T, define the longitudinal evolution of the strongly anisotropic phonon system.

Making a Galilean transformation to the coordinate frame (t', z'), which moves with velocity c along the axis z, and subtracting equation (30) from (29), we obtain the system of equations

$$\frac{\partial E}{c\partial t'} = -\frac{\partial}{\partial z'}(\alpha E), \tag{31}$$

$$E\frac{\partial \delta}{c\partial t'} = 0, \tag{32}$$

where we introduce α and δ which are small values of the same order of magnitude

$$\alpha = \langle (p\psi(p))' \rangle - \langle \zeta \rangle, \tag{33}$$

$$\delta = \langle \psi(p) \rangle + \langle \zeta \rangle, \tag{34}$$

and we have transformed equation (32) taking into account equation (31) and neglecting terms with higher powers of small parameters.

We may say that equation (31) defines the deformation of the energy density E in the frame moving with sound velocity c, and equation (32) determines the rate of change of this deformation.

We can estimate the characteristic time τ of the energy density deformation, by considering that all values have the same characteristic scale of spatial change along the z-axis, $l \sim ct_{\mathrm{p}}$. Then from the left- and right-hand side parts of equation (31) we obtain

$$\tau \sim \frac{1}{\alpha} t_{\mathrm{p}}. \tag{35}$$

For typical values $\alpha \sim 0.01$ and $t_{\mathrm{p}} \sim 10^{-6}$ s we find $\tau \sim 10^{-4}$ s. During this time, the ℓ-phonon pulse moving with the sound velocity $c = 238$ m s^{-1} travels a distance $l_{tr} \sim c\tau \sim 24$ mm.

We estimate the propagation distance over which the quadratic terms α^2, which were neglected in equation (31), can influence the shape of the pulse. We find that even for the shortest pulse duration, $t_{\mathrm{p}} \sim 10^{-7}$ s, $ct_{\mathrm{p}}/\alpha^2 \sim 240$ mm. This distance is much longer than the maximum distance used in experiments [6], so we can neglect such quadratic terms in equation (31).

We have a different situation with equation (32). In this case, including the quadratic terms in the rhs of equation (32) gives the value τ, equation (35), for the characteristic time for change of δ. Therefore the formal solution of equation (32), $\delta = \text{const}$, is only valid as long as the energy density and all the other values do not change significantly.

The initial conditions for the values χ and T are not known independently from the experiment [6]. So let us consider the simplest reasonable assumption that $\chi = \text{const}$ initially. Assuming that the initial profile of the energy density E is bell shaped, we obtain the same type of profile for the temperature T, because the dependence between T and E is monotonic.

Suppose χ is constant not only initially, but also at subsequent times. We will show that in the linear approximation with respect to the averages $\langle \psi(p) \rangle$ and $\langle \zeta \rangle$, equations (29) and (30) become the same. This will mean that equation (32) identically satisfies the approximation discussed above.

Denoting the partial derivative with respect to the temperature T, at $\chi = \text{const}$, by a prime, we rewrite the system of equations (29) and (30) as follows:

$$E'\frac{\partial T}{c\partial t} + \left[(1+\alpha)E' + \alpha' E\right]\frac{\partial T}{\partial z} = 0, \tag{36}$$

$$\left[(1-\delta)E' - \delta' E\right]\frac{\partial T}{c\partial t} + \left[(1+\alpha-\delta)E' + (\alpha' - \delta')E\right]\frac{\partial T}{\partial z} = 0. \tag{37}$$

If we divide by the factors at $\partial T/c\partial t$ in equations (36) and (37) we obtain, to the linear approximation with respect to the averages $\langle \psi(p) \rangle$ and $\langle \zeta \rangle$, the same equation:

$$\frac{\partial T}{\partial t} + v(T)\frac{\partial T}{\partial z} = 0, \tag{38}$$

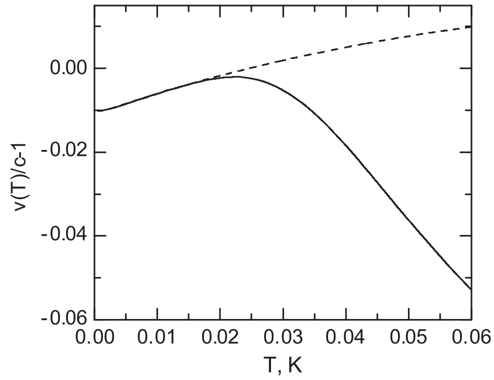

Figure 4. The dependence of the velocities $v(T)_{p_c^{(\infty)}}/c-1$ (solid line) and $v(T)_{p_c^{(2)}}/c-1$ (dashed line) on the temperature T, calculated at the typical value $\chi = 0.02$.

where we introduce the following notation:

$$v(T) = c\left[1+\alpha+\alpha'\frac{E}{E'}\right]. \tag{39}$$

Taking into account the initial condition for the temperature

$$T_{t=0} = T_0(z), \tag{40}$$

equation (38) determines the longitudinal evolution of the temperature and, correspondingly, that of the phonon energy density of the strongly anisotropic phonon system.

The solution of the equation (38), which satisfies the initial condition (40), is determined implicitly by the formula

$$T(z,t) = T_0(z - v(T)t). \tag{41}$$

From the expression (41) it follows that the value $v(T)$ determines the velocity with which every value of the temperature T moves along the z-axis.

By using equations (39), (33), (23) and (24) we can analyse the dependence of the velocity $v(T)$ on the temperature T. As follows from equations (24)–(26), for small values of T and for not very small χ we have

$$v(T) = c\left(1-\frac{\chi}{2}+\frac{27\gamma k_B^2T^2}{c^2(p_c^{(\infty)})^2\chi^2}\right). \tag{42}$$

Figure 4 shows the dependence of the velocity $v(T)$ on the temperature T, calculated at the typical value $\chi = 0.02$. The value $v(T)$ starts from the value $v = c(1-\chi/2)$ at $T = 0$. The decreasing value of the velocity $v(T)$ (figure 4, solid line) is due to the decreasing value of the function $\psi(p)$ at large momenta. This results in a negative contribution to the expression (39) described by equation (28), which reduces the velocity.

In contrast, if the phonon system is in local equilibrium with the distribution function (3) only for phonons with $p < p_c^{(2)}$, then the velocity $v(T)$ increases monotonically (figure 4, dashed line), because in this case the main contribution to the average $\langle(p\psi(p))'\rangle$ is determined by phonons in the middle of the momentum range, where $\psi(p)$ increases with increasing phonon momentum p.

The longitudinal evolution of the strongly anisotropic phonon pulse, defined by equation (41), depends on the behaviour of the function $v(T)$. If $v(T)$ increases monotonically with increasing T, then the front of the pulse steepens with time, and the back is drawn out.

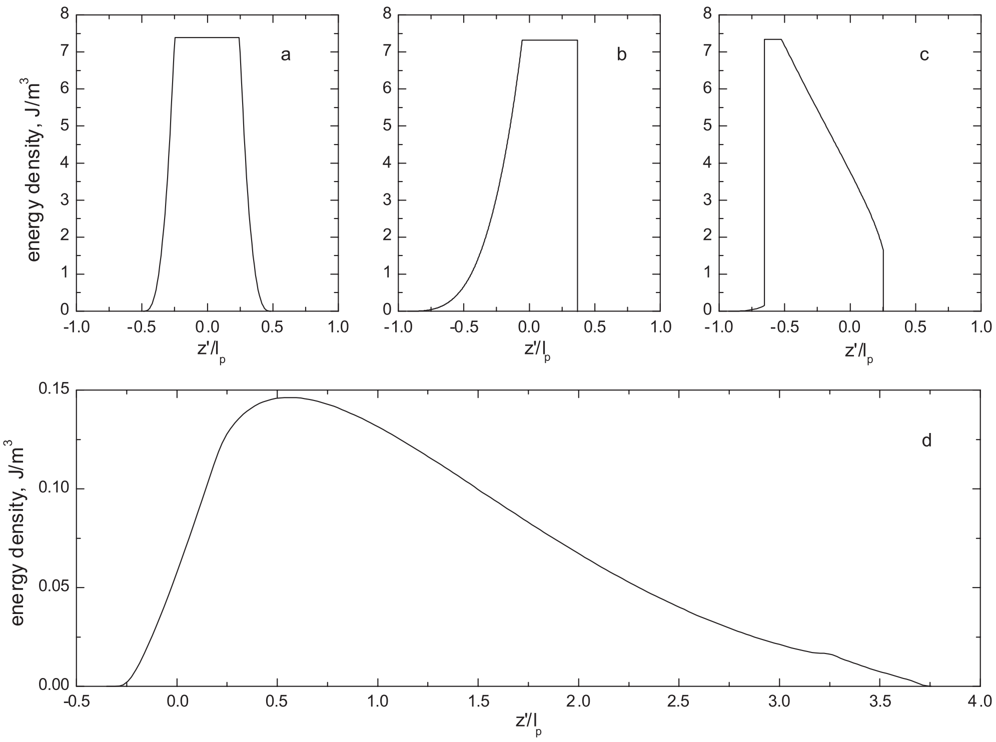

Figure 5. We show how the phonon pulse shape has evolved after propagating 10 mm, in the frame $z' = z - ct$ that moves relative to the laboratory frame z, with sound velocity c, under various conditions. The initial energy density is shown in (a). (b) shows the evolved energy density for the strongly interacting and very anisotropic phonon system, when phonons with momentum $p \sim< p_c$ are neglected. This corresponds to the monotonic velocity $v(T)$ (figure 4, dashed line). (c) shows the evolved energy density for the strongly interacting and very anisotropic phonon system, when phonons with momentum $p \sim< p_c$ are included. This corresponds to the non-monotonic velocity $v(T)$ (figure 4, solid line). (d) shows the evolved energy density for the pulse of a non-interacting phonon system that has the same initial distribution function as the strongly interacting phonon system. The initial pulse is cylindrical with radius 0.564 mm, to simulate the finite size of the heater. The numerical values are $c = 238$ m s^{-1}, $\chi = 0.02$ and $l_p = ct_p = 0.238$ mm.

This is realized in our case, when there are only phonons with $p < p_c^{(2)}$ characterized by the local equilibrium distribution function (3) (see figure 4, dashed line). This resembles the situation for the ordinary hydrodynamics; in the course of time, the front of the pulse breaks; the location of this is determined by the area rule.

If $v(T)$ depends non-monotonically on T, then a break can arise not only on the front, but also on the back of the phonon pulse. We have this situation, when all ℓ-phonons with $p < p_c^{(\infty)}$ are in equilibrium with the distribution function (3) (see figure 4, solid line).

In figure 5 we show calculations of the pulse shape after it has propagated away from the heater. The input pulse, for the calculations, is shown in figure 5(a). The width at half-height is 0.75 μs and the peak temperature is 0.04 K. A perfectly rectangular pulse would propagate without any change of shape. However it is unstable to perturbations as it propagates, so we have added a smooth rise and fall to the pulse to avoid infinitely steep sides.

In figure 5(b) we show the resulting pulse shape after 10 mm for the case when we ignore the phonons with momentum $p \sim< p_c$, but maintain the initial energy in the input pulse. We see that the leading edge has become vertical which is a result of the singular wave breaking. This is due to the velocity of the pulse increasing with amplitude as described by equation (39) and shown as the dashed line in figure 4(a). For pulses of smaller peak amplitude, the break occurs at longer propagation distances.

In figure 5(c) we show the pulse shape after 10 mm when we include the phonons with momentum $p \sim< p_c$. These will be present at $T = 0.04$ K according to the correct distribution function, equation (2). In this case we see that there are breaks at the back as well as the front of the pulse. This comes from the region of negative gradient of the velocity graph, as a function of temperature, shown as the solid line in figure 4. We see that the width of the pulse is almost exactly the width of the initial pulse. If the temperature amplitude of the pulse is less than 0.03 K, which is where the maximum velocity occurs, see figure 4, then the break at the back of the pulse cannot form. This is due to the relatively small number of phonons with momentum $p \sim< p_c$ at $T = 0.03$ K. In practice it may be hard to satisfy the condition that the temperature is low enough that no h-phonons are created, and that the temperature is high enough for there to be a relatively high number of phonons with $p \sim< p_c$.

To compare with the behaviour of these strongly interacting phonon pulses, we show the pulse shape for a similar pulse of non-interacting phonons in figure 5(d). This pulse mainly expands longitudinally due to phonon dispersion. The angular distribution of the initial pulse coupled with the finite size of the heater gives a relatively small contribution. We see that after 10 mm the pulse width is much larger than the initial pulse.

The important conclusion from these calculations is that phonon pulses show very little longitudinal expansion when the phonons are strongly interacting but expand considerably when the phonons do not interact. Furthermore it is clear from figures 5(b) and (c) that the strongly interacting pulses travel at the sound velocity.

5. Discussion

We have derived the system of equations which describes the evolution of a ℓ-phonon pulse in superfluid ^{4}He with the exact local equilibrium distribution function of the strongly anisotropic phonon system. For a phonon pulse, where we neglect the transverse deformation, we have obtained the equation for the longitudinal pulse shape, whose solution is a simple running wave. This solution describes the longitudinal deformation of the temperature of the phonon system under the condition of instant ℓ-phonon relaxation and when we neglect the phenomenon of h-phonon creation, i.e. an ℓ-phonon pulse at a low temperature.

Earlier, in the work [18], we considered the evolution of the ℓ-phonon system described by the cone distribution. This distribution has the main characteristics of the strongly anisotropic phonon systems, but it is not the exact local equilibrium distribution for the phonon collision integral. Moreover the cone distribution has a different momentum dependence (see figure 1). Nevertheless the two distribution functions give qualitatively the same results not only for the transverse evolution [18, 19], but also for the longitudinal deformation of the phonon pulse. Indeed, in [18] we have obtained the same type of equation differing only in the quantitative behaviour of the velocity with which each value of temperature moves. So, we can conclude as in [18] that in spite of rather large phonon dispersion, the pulse of strongly interacting phonons deforms rather slowly. The extent of the deformation is given by difference $v(T_{\max}) - c$, where $T_{\max}$ is the maximum temperature of the pulse, while the phonon pulse formed by weakly interacting phonons would deform much more strongly [20].

At present, it is not possible with the existing experimental techniques [6] to measure the longitudinal structure of ℓ-phonon pulses. The observation of the pulse shapes in figures 5(b) and (c) would make it possible to decide whether phonons with momentum $p \sim< p_c$

significantly contribute or not. However, for the non-interacting phonon pulses, the dispersion would result in an easily detectable longitudinal broadening, as is seen at high pressures where the dispersion is normal and consequently there are no 3pp interactions [21]. As this is not seen at zero pressure, we may say that the experiment [6] confirms that ℓ-phonons interact strongly.

New experimental results, concerning the longitudinal structure of the ℓ-phonon pulses, would help to answer the question of whether all phonons with momenta $p < p_c^{(\infty)}$, or only phonons with $p < p_c^{(2)}$, are in equilibrium. Our theory predicts essentially different deformations of the ℓ-phonon pulses in these two cases.

Acknowledgments

This research was supported by EPSRC of the UK (grant EP/C 523199/1), and by GFFI of Ukraine (grant N 02.07/000372).

References

[1] Stirling W G 1983 *75th Jubilee Conf. on Liquid Helium-4* ed J G M Armitage (Singapore: World Scientific) p 109
Stirling W G 1998 private communication
[2] Landau L D 1947 *J. Phys. USSR* **11** 91
[3] Tucker M A H and Wyatt A F G 1994 *J. Phys.: Condens. Matter* **6** 2813
[4] Tucker M A H and Wyatt A F G 1994 *J. Phys.: Condens. Matter* **6** 2825
[5] Tucker M A H and Wyatt A F G 1998 *J. Low Temp. Phys.* **113** 621
[6] Vovk R, Williams C D H and Wyatt A F G 2003 *Phys. Rev.* B **68** 134508
[7] Maris H J and Massey W E 1970 *Phys. Rev. Lett.* **25** 1220
[8] Jäckle J and Kerr K W 1971 *Phys. Rev. Lett.* **27** 654
[9] Wyatt A F G, Lockerbie N A and Sherlock R A 1989 *J. Phys.: Condens. Matter* **1** 3507
[10] Dynes R C and Narayanamurti V 1974 *Phys. Rev. Lett.* **33** 1195
[11] Adamenko I N, Kitsenko Y u A, Nemchenko K E, Slipko V A and Wyatt A F G 2005 Three phonon scattering in isotropic and anisotropic phonon systems at different pressures *Low Temp. Phys.* **31** at press
[12] Maris H J 1977 *Rev. Mod. Phys.* **49** 341
[13] Mezei F, Lartigue C and Farago B 1991 *Excitations in 2-Dimensional and 3-Dimensional Quantum Fluids (Exeter, 1990)* ed A F G Wyatt and H J Lauter (New York: Plenum) p 119
[14] Adamenko I N, Nemchenko K E and Wyatt A F G 2002 *J. Low Temp. Phys.* **126** 1471
[15] Adamenko I N, Nemchenko K E, Zhukov A V, Tucker M A H and Wyatt A F G 1999 *Phys. Rev. Lett.* **82** 1482
[16] Wyatt A F G, Tucker M A H, Adamenko I N, Nemchenko K E and Zhukov A V 2000 *Phys. Rev.* B **62** 9402
[17] Andreev A F and Melnikovsky L A 2003 *JETP Lett.* **78** 574
[18] Adamenko I N, Nemchenko K E, Slipko V A and Wyatt A F G 2003 *Phys. Rev.* B **68** 134507
[19] Adamenko I N, Nemchenko K E, Slipko V A and Wyatt A F G 2005 Transverse evolution of cylindrical phonon pulse in superfluid helium-4 *J. Low Temp. Phys.* at press
[20] Adamenko I N, Nemchenko K E and Wyatt A F G 2004 *Low Temp. Phys.* **30** 432
[21] Wyatt A F G, Sherlock R A and Allum D R 1982 *J. Phys. C: Solid State Phys.* **15** 1897